MATHEMATICS

1_. $S = \sum_{K=1}^{2006} \frac{K+2}{K!+(K+1)!+(K+2)!} = \sum_{K=1}^{2006} \frac{K+2}{K!(K+2)^2} = \sum_{K=1}^{2006} \frac{1}{K!(K+2)}$

$= \sum_{K=1}^{2006} \frac{K+1}{(K+2)!} = \sum_{K=1}^{2006} \frac{K+2-1}{(K+2)!} = \sum_{K=1}^{2006}\left[\frac{1}{(K+1)!} - \frac{1}{(K+2)!}\right] = \frac{1}{2} - \frac{1}{2008!}$

2_. $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots \infty = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$

3_. $xA = yG \quad \Rightarrow \quad \frac{x}{y} = \frac{G}{A} = \frac{2\sqrt{ab}}{a+b}$

$yG = zH \quad \Rightarrow \quad \frac{y}{z} = \frac{2\sqrt{ab}}{a+b} \quad \therefore \quad \frac{x}{y} = \frac{y}{z}$

4_. $(1 - 2x + 2x^2)^{743}(2 + 3x - 4x^2)^{744} = a_0 + a_1x + \ldots + a_{2974}x^{2974}$

Put $x = 1 \quad \Rightarrow \quad 1 = a_0 + a_1 + \ldots + a_{2974}$

5_. Let $\alpha_n = (2 + \sqrt{3})^n = I + f$ where $0 < f < 1$

Let $G = (2 - \sqrt{3})^n \quad \Rightarrow \quad I + f + G = 2[{}^nC_0\, 2^n + {}^nC_2 . 2^{n-2} . 3 + \ldots] \Rightarrow \quad f + G$ is integer

But $0 < f + G < 2 \quad \Rightarrow \quad f + G = 1$

$\therefore \quad \alpha_n - [\alpha_n] = f = 1 - G = 1 - (2 - \sqrt{3})^n \quad \Rightarrow \quad \lim_{n\to\infty} (\alpha_n - [\alpha_n]) = 1 - 0 = 1$

6_. 6, 12, 18, ... , 294 $\Rightarrow$ 49 numbers

18, 36, 54, ... , 288 $\Rightarrow$ 16 numbers $\quad \therefore \quad 49 - 16 = 33$

7_. Let $E = \frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_1} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_1} + \frac{a_3}{a_2} + \frac{a_3}{a_4} + \frac{a_4}{a_1} + \frac{a_4}{a_2} + \frac{a_4}{a_3}$

A.M. $\geq$ G.M. $\quad \Rightarrow \quad \frac{E}{12} \geq \left(\frac{a_1}{a_2}.\frac{a_1}{a_3}. \ldots\ldots \frac{a_4}{a_3}\right)^{1/12} \quad \Rightarrow \quad E \geq 12$

8. $\sum_{n=2}^{\infty} \frac{U_n}{U_{n-1}U_{n+1}} = \sum_{n=2}^{\infty} \frac{U_{n+1} - U_{n-1}}{U_{n-1}U_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{U_{n-1}} - \frac{1}{U_{n+1}}\right) = \frac{1}{U_1} + \frac{1}{U_2} = 2$

9. $\frac{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\ldots\text{ n terms}}{\left(1-\frac{1}{3}\right)} = \frac{3}{2}\left[1 - \frac{1}{3^{2^n}}\right] = \frac{3}{2}$ as $n \to \infty$

10. $S_n - S_{n-2} = 2$ (for odd $n \geq 3$)

$\Rightarrow \quad T_n + T_{n-1} = 2 \quad \Rightarrow \quad \left(\frac{1}{n^2} + 1\right)T_{n-1} = 2 \Rightarrow T_{n-1} = \frac{2n^2}{1+n^2} \quad \Rightarrow \quad T_m = \frac{2(m+1)^2}{1+(m+1)^2}$

11. $(19-4)^{23} + (19+4)^{23} = 2[{}^{23}C_0\, 19^{23} 4^{\circ} + \ldots + {}^{23}C_{22}\, 19^1 4^{22}]$

12. $\sum_{r=0}^{20} r(20-r).{}^{20}C_r.{}^{20}C_r = \sum_{r=0}^{19} r(20-r)^{20} C_r {}^{20}C_{20-r} = 400\sum_{r=0}^{19} {}^{19}C_{r-1}.{}^{19}C_{19-r} = 400 . {}^{38}C_{18} = 400 . {}^{38}C_{20}$

13. $\left(1+\sqrt{x}+\frac{1}{\sqrt{x}-1}\right)^{-30} = \left(\frac{\sqrt{x}-1}{x}\right)^{30} = \frac{\left(\sqrt{x}-1\right)^{30}}{x^{30}}$ $\Rightarrow$ these is no constant term

14_*. $b = {}^{20}C_0 + {}^{20}C_1 + \ldots . + {}^{20}C_9 = {}^{20}C_{20} + \ldots . + {}^{20}C_{11} = c$

$\Rightarrow$ $a = b + c + {}^{20}C_{10}$ $\Rightarrow$ $a = 2b + {}^{20}C_{10}$

$\Rightarrow$ $a - 2b = \frac{20!}{10!10!} = \frac{(2.4......20)(1.3.5....19)}{10!10!} = \frac{2^{10}(1.3.5.....19)}{10!}$

15_*. $z = 2x + 4y$, $xy = 48$, $\frac{2xy}{x+y} = 6$ $\Rightarrow$ $x + y = 16$ $\Rightarrow$ $x = 12, y = 4$

$\Rightarrow$ $z = 24 + 16 = 40$ $\Rightarrow$ $p = 5$

When $-1 < x < 1$ then $\frac{12x}{x^2+1} \in [-6, 6]$

$\theta \in \left(0,\frac{\pi}{6}\right)\cup\left(\pi,\frac{3\pi}{2}\right)$ $\Rightarrow$ $\cos\theta \in (-1, 0) \cup \left(\frac{\sqrt{3}}{1},1\right)$ $\Rightarrow$ $5 + \cos\theta \in (4, 5) \cup \left(5+\frac{\sqrt{3}}{2},6\right)$

16_*. 1 2 . . . . 9 $\Rightarrow$ 9

10 11 . . . . 99 $\Rightarrow$ 180

190^{th} digit is1 ($\therefore$ 100)

201^{st} digit is 3 (100 101 102 103)

100 101 102 . . . . 707 $\Rightarrow$ $608 \times 3 = 1824$ $\Rightarrow$ $9 + 180 + 1824 = 2013$

so 2014^{th} digit is 7. ($\because$ 708)

17_*. $7^{2014} = 49(1 + 2400)^{503} = 49(1207201 + 10^4\lambda) = 59152849 + 10^4K$

Divisors are $7^0, 7^1, 7^2, \ldots, 7^{2014}$

$\Rightarrow$ No. of divisors are 2015, composite divisors 2013 and prime divisors 1 $\Rightarrow$ $p = 1$

Also no of ways to express a non-zero vector coplanar with two given non-collinear vectors as a linear combination of the two vectors = 1

18_*. $2\alpha_{r+2} = \alpha_r + \alpha_{r+1}$

$\Rightarrow$ $2(\alpha_{r+2} - \alpha_{r+1}) = \alpha_r - \alpha_{r+1}$ $\Rightarrow$ $\alpha_{r+2} - \alpha_{r+1} = -\frac{1}{2}(\alpha_{r+1} - \alpha_r)$

$\Rightarrow$ $\alpha_{10} - \alpha_9 = -\frac{1}{2}(\alpha_9 - \alpha_8) = \frac{1}{4}(\alpha_8 - \alpha_7) = \ldots . = -\frac{1}{2^9}(\alpha_1 - \alpha_0) = \frac{-16}{2^9} = \frac{-1}{32}$

As $\alpha_0 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3,$ are in G.P.

$\Rightarrow$ $\alpha_0 - \alpha_2, 2(\alpha_1 - \alpha_2), \alpha_1 - \alpha_3$ are in H.P.(Adding middle term to all terms)

19_*. $2K A_K = (2K - 3)A_{K-1}$ $\Rightarrow$ $2K A_K - 2(K - 1)A_{K-1} = -A_{K-1}$

put $K = 2, 3, 4, 5, \ldots$

$\Rightarrow$ $4A_2 - 2A_1 = -A_1$

$6A_3 - 4A_2 = -A_2$

....

$2KA_K - 2(K - 1)A_{K-1} = -A_{K-1}$

$\Rightarrow$ $2KA_K - 2A_1 = -(A_1 + \ldots + A_{K-1})$ $\Rightarrow$ $A_1 + A_2 + \ldots + A_k = 1 - (2k - 1)A_k$

As $(2K - 1) A_K > 0$ $\Rightarrow$ $A_1 + A_2 + \ldots + A_k < 1$ where $k \geq 2$

20_*. $A_m = a + m\left(\frac{2b-a}{n+1}\right)$

$A_m' = 2a + m\left(\frac{b-2a}{n+1}\right)$

$\Rightarrow \quad a(n + 1) + m(2b - a) = 2a(n + 1) + m(b - 2a)$

$\Rightarrow \quad bm = a(n - m + 1) \quad \Rightarrow \quad \frac{a}{b} < n \quad \Rightarrow \quad m < n^2 - mn + n$

$\Rightarrow \quad m - n < n(n - m)$ which is false for n = m

$\frac{a}{b} \le m \quad \Rightarrow \quad \frac{m}{n-m+1} \le m \quad \Rightarrow \quad 0 \le m(n - n)$ which is true.

21_. $\frac{b-c}{a-b} = \frac{[A+(q-1)D]-[A+(r-1)D]}{[A+(p-1)D]-[A+(q-1)D]} = \frac{q-r}{p-q}$ Rational Number

22_*. $t_n = \frac{n^2+n-1}{(n+2)!} = \frac{(n^2+2n)-(n+1)}{(n+2)!} = \frac{n}{(n+1)!} - \frac{n+1}{(n+2)!} = \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) - \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right)$

$S_n = \left(1 - \frac{1}{(n+1)!}\right) - \left(\frac{1}{2} - \frac{1}{(n+2)!}\right) = \frac{1}{2} - \frac{1}{(n+1)!} + \frac{1}{(n+2)!}$

23_*. $a_n = \frac{1000}{1}.\frac{1000}{2}......\frac{1000}{1000}.\frac{1000}{1001}.\frac{1000}{1002}......\frac{1000}{n}$, n > 1000

$\Rightarrow \quad a_n \to 0$ as $n \to \infty$

$a_n = a_{n+1} \quad \Rightarrow \quad \frac{1000^n}{n!} = \frac{1000^{n+1}}{(n+1)!} \quad \Rightarrow \quad n + 1 = 1000 \quad \Rightarrow \quad n = 999.$

24_*. $b_K = \frac{n}{2}[a_K + a_{K+n-1}] = \frac{n}{2}[a_1 + (K - 1)d + a_1 + (n + K - 2)d]$

$= \frac{n}{2}[2a_1 + (K - 1)d + (n - 1)d + (K - 1)d] = \frac{n}{2}[a_n + a_1 + 2(K - 1)d]$

$\sum_{K=1}^{n} b_K = \frac{n}{2}\left[na_n + na_1 + 2d\frac{n(n-1)}{2}\right] = \frac{n^2}{2}\left[a_n + a_1 + d(n-1)\right] = n^2 a_n$

25_*. $f(n) = {}^nC_0({}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1})$

$+ {}^nC_1({}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1})$

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$+ {}^nC_n\, {}^{n+1}C_{n+1}$

$= ({}^nC_0\, {}^{n+1}C_1 + {}^nC_1\, {}^{n+1}C_2 + \ldots + {}^nC_n\, {}^{n+1}C_{n+1})$

$+ ({}^nC_0\, {}^{n+1}C_2 + {}^nC_1\, {}^{n+1}C_3 + \ldots + {}^nC_{n-1}\, {}^{n+1}C_{n+1})$

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$= {}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \ldots + {}^{2n+1}C_0$

$= 2^{2n}$

26_*. $(1 - x)^n (1 + x + x^2)^n = ({}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \ldots)(a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n})$

$\Rightarrow \quad (1 - x^3)^n = ({}^nC_0\, a_r - {}^nC_1\, a_{r-1} + \ldots)\, x^r$ + other terms

Required expression = coefficient of x^r in ${}^nC_0 - {}^nC_1 x^3 + {}^nC_2x^6 - \ldots = 0$ if $r \ne 3\lambda$ and $(-1)^{r/3}\, {}^nC_{r/3}$ if $r = 3\lambda$

27_. $2({}^{26}C_0 + {}^{26}C_1 + \ldots + {}^{26}C_{13}) = ({}^{26}C_0 + \ldots + {}^{26}C_{26}) + {}^{26}C_{13} = 2^{26} + {}^{26}C_{13}$

$(1 - 3)^{25} = {}^{25}C_0 - {}^{25}C_1\, 3^1 + \ldots - {}^{25}C_{25}\, 3^{25}$

28_*. ${}^{100}C_6 + {}^{100}C_7 + 3({}^{100}C_7 + {}^{100}C_8) + 3({}^{100}C_8 + {}^{100}C_9) + {}^{100}C_9 + {}^{100}C_{10} = {}^{101}C_7 + 3({}^{101}C_8) + 3({}^{101}C_9) + {}^{101}C_{10}$

$= {}^{101}C_7 + {}^{101}C_8 + 2({}^{101}C_8 + {}^{101}C_9) + {}^{101}C_9 + {}^{101}C_{10} = {}^{102}C_8 + 2.{}^{102}C_9 + {}^{102}C_{10}$

$= {}^{103}C_9 + {}^{103}C_{10} = {}^{104}C_{10} \quad \Rightarrow \quad {}^xC_y = {}^{104}C_{10}$ or ${}^{104}C_{94}$

29^. Put x = 1 & –1 and add $4^{20} + 4^{20} = 2(a_0 + a_2 + \ldots + a_{60})$

Now subtract $\Rightarrow \quad 0 = 2(a_1 + a_3 + \ldots + a_{59})$

$a_0 = 2^{20}$ and a_{59} = coeff of x^{59} in $(2 - 3x + 2x^2 + 3x^3)^{20} = {}^{20}C_1.2.3^{19}$

30. $a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots\ldots = (1 + 2x^2 + x^4)(1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots\ldots\ldots)$

$= 1 + {}^nC_1 x + (2 + {}^nC_2)x^2 + (2\,{}^nC_1 + {}^nC_3)x^3 + \ldots\ldots\ldots$

Now $2a_2 = a_1 + a_3$

for n = 2 we have $a_1 = 2$, $a_2 = 3$, $a_3 = 4$ which are in A.P.

for $n \geq 3$ we have $2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1) \Rightarrow n^3 - 9n^2 + 26n - 24 = 0 \Rightarrow n = 2, 3, 4 \Rightarrow n = 3, 4$

31. $f(m) = \sum_{r=0}^{m} {}^{30}C_{30-r}\ {}^{20}C_{m-r} = \sum_{r=0}^{m} {}^{30}C_r\ {}^{20}C_{m-r} \Rightarrow f(m) = {}^{50}C_m$

$f(33) = {}^{50}C_{33} = {}^{50}C_{17} = \dfrac{34.35.36\ldots\ldots..50}{17!}$ which is multiple of 37

32. ${}^{15}C_1 + {}^{16}C_2 + {}^{17}C_3 + \ldots\ldots + {}^{39}C_{25} = {}^{15}C_0 + {}^{15}C_1 + {}^{16}C_2 + {}^{17}C_3 + \ldots\ldots + {}^{39}C_{25} - {}^{15}C_0 = {}^{40}C_{25} - 1$

33. $\left(8 + 3\sqrt{7}\right)^n = I + f$; $\left(8 - 3\sqrt{7}\right)^n = f'$

Adding $I + f + f' = 2$ (integer) $\Rightarrow$ $f + f'$ = integer $\Rightarrow$ $f + f' = 1$

34. $(1 - y)^m (1 + y)^n = 1 + (n - m)y + \left\{\dfrac{m(m-1)}{2} + n\left(\dfrac{n-1}{2}\right) - mn\right\} y^2 \ldots\ldots.$

Also $a_1 = a_2 = 10$ $\Rightarrow$ $n - m = 10$ & $a_2 = \dfrac{m^2 + n^2 - m - n - 2mn}{2} = 10$

$\Rightarrow$ $(m - n)^2 - (m + n) = 20$ $\Rightarrow$ $m + n = 80$ $\Rightarrow$ $m = 35$ $\Rightarrow$ $n = 45$

Sol. (35 to 37)

$f(n) = n^2 + 1,\ g(n) = n^2 + n \Rightarrow \lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 1$

$F(n) = \sum(n^2 + 1) = \dfrac{n}{6}(2n^2 + 3n + 7)$

$G(n) = \sum(n^2 + n) = \dfrac{n(n+1)(n+2)}{3} \Rightarrow \lim_{n\to\infty} \dfrac{F(n)}{G(n)} = 1$

$$\lim_{n\to\infty}\left(\frac{F(n)}{G(n)}\right)^n - \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)^n = \lim_{n\to\infty}\left(\frac{2n^3 + 3n^2 + 7n}{6} \times \frac{3}{n^3 + 3n^2 + 2n}\right)^n - \lim_{n\to\infty}\left(\frac{n^2 + 1}{n^2 + n}\right)^n$$

$$= e^{\lim_{n\to\infty}\frac{\left(-3n^2 + 3n\right)n}{n\left(2n^2 + 6n + 4\right)}} - e^{\lim_{n\to\infty}\left(\frac{n^2 + 1 - n^2 - n}{n^2 + n}\right)n} = e^{-3/2} - e^{-1}$$

38. $S = \sum_{r=1}^{\infty} \dfrac{r + 2}{2^{r+1}\ .\ r\ (r+1)} = \sum\left(\dfrac{2}{r} - \dfrac{1}{r+1}\right)\dfrac{1}{2^{r+1}} = \sum\left(\dfrac{1}{r.2^r} - \dfrac{1}{(r+1)2^{r+1}}\right) = 1/2$

39. $S = 1 + \dfrac{4}{3} + \dfrac{9}{9} + \dfrac{16}{27} + \ldots\ldots\infty$

$\dfrac{1}{3}S = \dfrac{1}{3} + \dfrac{4}{9} + \dfrac{9}{27} + \ldots\ldots\infty \Rightarrow \dfrac{2}{3}S = 1 + \dfrac{3}{3} + \dfrac{5}{9} + \dfrac{7}{27} + \ldots\ldots\ldots\infty \Rightarrow \dfrac{2}{3}S = 3$

40. $E = \lim_{n\to\infty} \sum_{r=1}^{n}\left(\sum_{t=0}^{r-1} \dfrac{1}{5^n}\,{}^nC_r\,{}^rC_t\,3^t\right) = \lim_{n\to\infty}\sum_{r=1}^{n} \dfrac{{}^nC_r}{5^n}(4^r - 3^r) = \lim_{n\to\infty}\left(\dfrac{(5^n - 1)}{5^n} - \dfrac{(4^n - 1)}{5^n}\right) = 1 - 0 = 1.$